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Blocking Probability for a Multistage Clos Connecting Network

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The blocking probability is computed by assuming a thermodynamic limit when the number of stages increases above a certain value. In this limit we exhibit a set of two algebraic equations which gives the blocking probability as a function of the traffic demand. A comparison with a computer simulation of the system gives an excellent agreement.

KEY WORDS: Clos connecting network; blocking probability; computer simulation; system analysis.

1. INTRODUCTION

In this paper, we consider the application of the thermodynamic concepts to the study of connecting networks such as those found in telephone systems as illustrated in Fig. 1.

Basically, the problem is to obtain all possible connections between N inputs and N outputs given by the permutation group by means of a physical structure. This could, for example, be an $N \times N$ matrix. For telephone applications in which the value of N can reach several thousands, this hardware assembly, requiring N^2 crosspoints, is obviously quite difficult to make. The technological solution, therefore, generally consists in breaking this large matrix into much smaller ones linked in such a way to enable the realization of all connections between inputs and outputs.

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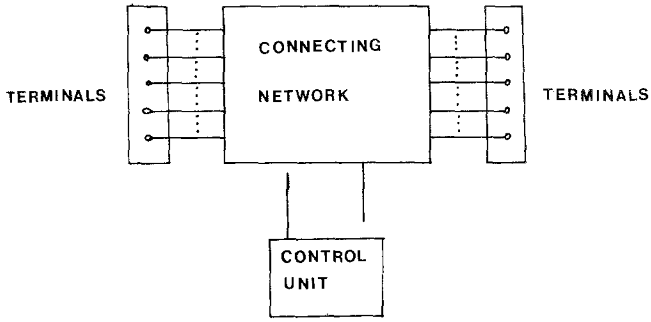

Fig. 1. Connecting systems.

The main feature of such a system is that both the connecting network and the control unit contain thousands of components giving a large number of possible combinations (microscopic states).

Considering the number of different possible configurations, $A_N = \sum_{i=0}^{N} (C_i^N)^2 i!$, the minimal number of (binary) components required is asymptotically equal to $C_N = N \log N - 1.443N$ $(N \to \infty)$.[1] Unfortunately, this ideal structure is unknown and we have to approximate this lower bound by the use of blocking networks where a new demand for connection is refused if there is not at least one free path between the input and the output concerned.

Our problem is to compute the probability of such an event (blocking probability) for a given level of the offered traffic. We recognize the similarity of this problem with transport problems in random media, e.g., percolation. However, an important difference lies in the possibility to provide the system with a certain intelligence. The control unit has a detailed knowledge of the evolution of the system state and can act in consequence.[2,3]

Two procedures are possible:

(a) The implementation of nonrandom routing policies which consist in a systematic choice among all possible free paths so that the probability of future blocking configurations is minimized.

(b) The implementation of algorithms which rearrange the calls giving rise to a blocking configuration in order to accept the blocked demand.

In this paper, we shall leave aside this important feature to consider the system in random selection mode whereby a path is chosen at random among those which are free. This random strategy provides an upper bound to the blocking probability or, in other words, it qualifies the worst behavior of the system.

The blocking probability is a principal measure of network performance and much study has gone into its estimation.[4-8]

We propose a new theoretic approach of the problem, for multistage Clos networks, based on statistical mechanic concepts such as the *thermodynamic limit*.

This approach has given a very good agreement with computer simulation results[2] for three-stage networks.

To obtain the multistage network, we start from a three-stage network and replace each central matrix itself by a three-stage network, thus obtaining a five-stage network. We repeat this operation until we obtain an assembly of identical components (matrices of size $n \times n$) arranged in $(2K + 1)$ stages (see Fig. 2).

It is interesting to note that such networks repeat their structure at a smaller scale where the basic element is still a three-stage network.

In fact the central idea will be the invariance of the network blocking probability (a macroscopic quantity) with respect to its size. Of course this property is true only in the limit $K \to \infty$.

One can see that for $n = 2$, the number of 2×2 switches is equal to $(N \log N - 0.5N)$ and which is very close to the ideal lower bound C_N.

The homogeneity of the hardware used, the optimal number of components, and the larger combinatorial power are the important properties of this class of networks which have long been an important feature of telephone exchanges. Moreover, these structures are giving rise to considerable interest in array computer architecture[9] for connecting memory blocks to processing elements.

2. THERMODYNAMIC LIMIT

In the thermodynamic limit, according to a particular prescription, we let the system become sufficiently large to be able to neglect states which occur with an insignificant probability. This means practically that above a critical size, by, say, doubling the system we do not change its average behavior. We shall call this property *extensivity*.[2]

For the three-stage Clos network defined in Fig. 3, the extensivity of the structure allows us to give an expression of the blocking probability independent of the parameters N and b[2,10]

$$p_B(n, t_E) = \frac{n-1}{n} t_E^n (2 - t_E)^{n-2} \tag{1}$$

where the Bernoulli simultaneous connection distribution for inlet and outlet matrices is assumed, t_E being the traffic loaded per inlet line by the use of a random routing strategy: $0 \leqslant t_E \leqslant 1$.

In multistage systems, the thermodynamic limit is reached by increasing k, keeping constant n in such a way that to obtain the next structure (with a number of inlet and outlet lines multiplied by n) we have just to

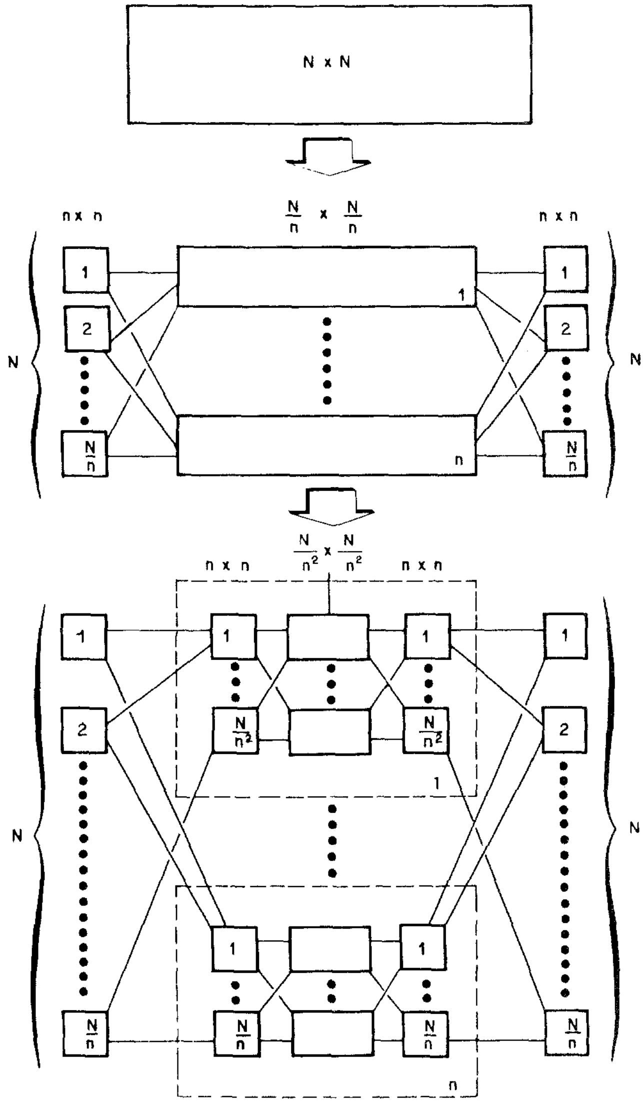

Fig. 2. Recurrent construction of a multistage Clos network.

consider n identical former structures and add as first and last stage two columns of matrices of size $n \times n$ (Fig. 2).

The computer simulations confirm that, following the mentioned prescription, for a given traffic, the blocking probability tends to a constant (Fig. 4), so that we can assume that above a critical value of k we have

$$P_B(n, t_E) = P_B(n, k, t_E) = P_B(n, k-1, t_E) \tag{2}$$

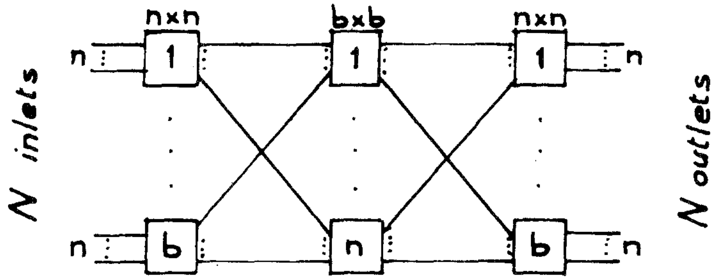

Fig. 3. Three-stage Clos network; $N = n \cdot b$.

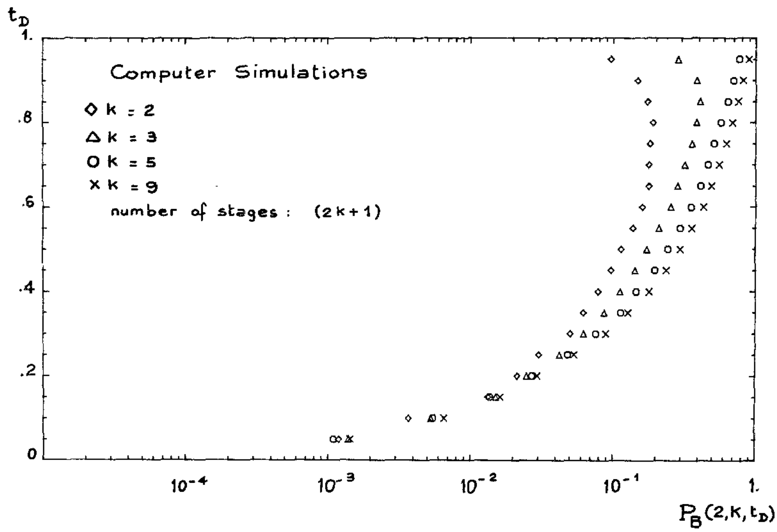

Fig. 4. Concept of thermodynamic limit; computer simulations; $N = n^{k+1}$, $n = 2$.

This relation of extensivity remains true for any routing procedures; however, we shall exhibit an explicit expression of (2) only for the random hunting strategy. The computer simulations are designed according to a Bernoullian-offered traffic process with parameter λ, μ and source occupancy t_D (average percentage of connected inlets if there were no blockings)

$$t_D = (1 + \mu/\lambda)^{-1} \tag{3}$$

To compute t_E in statistical equilibrium we must note that the mean rate of new connections equals the mean rate of terminations

$$(1 - P_B)\lambda(1 - t_E) = \mu t_E$$

and finally we get

$$t_E = \frac{(1 - P_B)t_D}{1 - P_B t_D} \tag{4}$$

Expression (4) allows us to display (2) as a function of the traffic demand t_D.

3. COMPUTATION OF THE BLOCKING PROBABILITY

It is important to consider a $(2k + 1)$-stage Clos network as a three-stage one, where central matrices are *independent blocking subsystems* with $2k - 1$ stages (Fig. 2). We represent this feature of the system in Fig. 5, where the pair of inlet and outlet matrices (i, j) characterizing the new demand of connection are linked by all possible paths through the n central subsystems.

Two situations lead to a blocking state, as shown in Fig. 6.

(a) *Mismatch due to the recursive three-stage Clos structure*: any central subsystem has at least one input or output link already busy.

In a real three-stage Clos network the central subsystems are strictly nonblocking and (a) is the only situation to be considered.

Now these subsystems have a blocking probability and we must take into account the following new situation.

(b) *At least one subsystem has idle input and output links.* We consider the set of such subsystems and the demand of connection will not be filled if none of the elements of this set cannot accomodate it.

In the assumption of random routing strategy, the extensivity relation (2) means that in the thermodynamic limit the blocking probability of the global system equals the blocking probability of any central subsystem, so

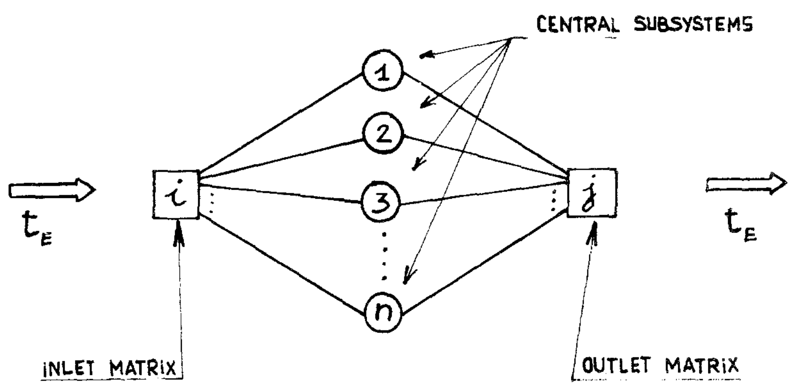

Fig. 5. Multistage Clos network; graphical representation.

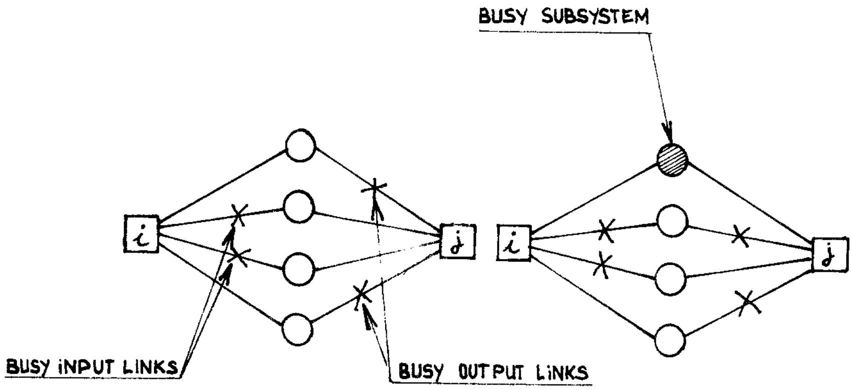

Fig. 6. (a) Mismatch due to the recursive three-stage structure. (b) Blocked central subsystem. Although the input–output link is idle, the hachured subsystem is in a blocking state.

that in agreement with (a) and (b), we write

$$P_B(n,t_E) = p_B(n,t_E) + \left[1 - p_B(n,t_E)\right]G(n,P_B,t_E) \tag{5}$$

where, assuming the Bernoulli's offered traffic, $p_B(n,t_E)$ is given by (1). $G(n,P_B,t_E)$ is the conditional probability of not being able to connect matrices i and j, through one of the central subsystems, on account of a blocking situation described in (b):

$$G = \sum_{k=0}^{n-1} \sum_{m=0}^{n-1} q_k(t_E)A_{k,m}(P_B)q_m(t_E) \tag{6}$$

where $q_k(t_E)$ is the probability to find k busy input (or output) links:

$$q_k = \binom{k}{n-1} t_E^k(1-t_E)^{n-1-k} \tag{8}$$

and $A_{k,m}$ the probability to find at least one subsystem satisfying (b), when k input links and m output links are busy.

Hence the problem will be to evaluate $A_{k,m}$, which in our assumptions is such that

$$A_{k,m} = A_{m,k} \tag{8}$$

Example: $n = 2$. We have to determine three terms (Fig. 7):

(i) $A_{0,0}$, the probability to be blocked even if all the input and output links are idle,

$$A_{0,0} = P_B^2$$

(ii) $A_{1,0}$, one input link is busy,

$$A_{1,0} = P_B$$

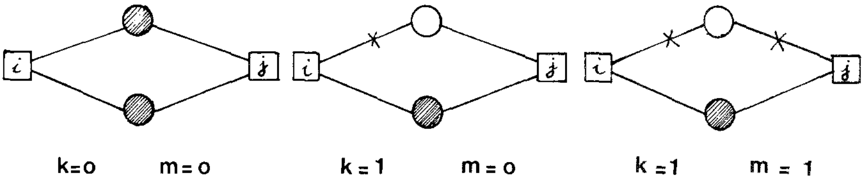

Fig. 7. Blocking configurations described in Fig. 6b. $n = 2$, k inputs links and m output links are busy.

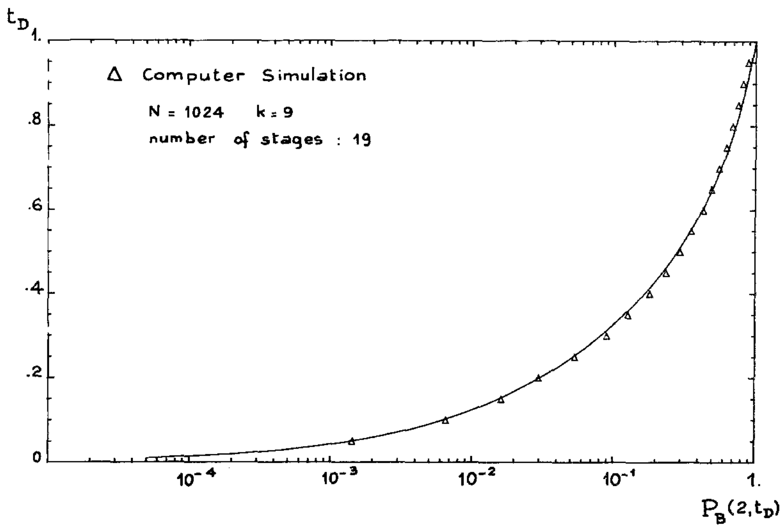

Fig. 8. Blocking probability in the thermodynamic limit. Comparison between the theory and the computer simulations, $n = 2$.

(iii) $A_{1,1}$, one input link and one output link are busy,

$$A_{1,1} = P_B$$

Introducing these values in (6) and (5) we get

$$P_B = \frac{t_E^2}{2} + \left(1 - \frac{t_E^2}{2}\right)\left[(1 - t_E)^2 P_B^2 + 2t_E(1 - t_E)P_B + t_E^2 P_B\right] \tag{9}$$

Combining (9) and (4) we obtain numerically $P_B(2, t_D)$. Figure 8 shows the agreement between this solution and the values measured from computer simulations.

Example: $n = 4$. We have to determine ten terms of $A_{k,m}$. It is easily verified that (see Fig. 9)

$$A_{k,0} = P_B^{4-k}, \qquad k = 0, 1, 2, 3$$

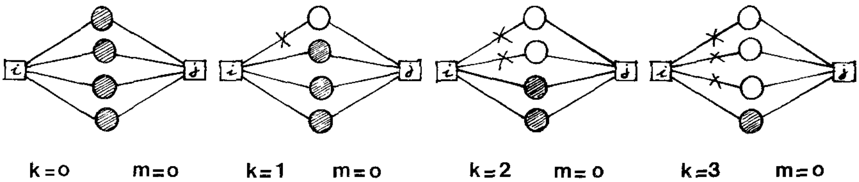

Fig. 9. Blocking configurations described in Fig. 6b. $n = 4$, k input links and m output links are busy.

For $A_{1,1}$, we have four possible configurations (Fig. 10), so we obtain

$$A_{1,1} = \frac{(P_B^3 + 3P_B^2)}{4}$$

For $A_{2,1}$, also four possible configurations (Fig. 11):

$$A_{2,1} = \frac{(P_B^2 + P_B)}{2}$$

For $A_{2,2}$, five possible configurations (Fig. 12):

$$A_{2,2} = \frac{(P_B^2 + 4P_B)}{5}$$

For $A_{3,1}$, three possible configurations (Fig. 13):

$$A_{3,1} = P_B$$

For $A_{3,2}$, also three possible configurations (Fig. 14):

$$A_{3,2} = P_B$$

For $A_{3,3}$, only one configuration (Fig. 15):

$$A_{3,3} = P_B$$

To summarize,

$$A_{k,m} = \begin{bmatrix} P_B^4 & P_B^3 & P_B^2 & P_B \\ P_B^3 & \dfrac{P_B^3 + 3P_B^2}{4} & \dfrac{P_B^2 + P_B}{2} & P_B \\ P_B^2 & \dfrac{P_B^2 + P_B}{2} & \dfrac{P_B^2 + 4P_B}{5} & P_B \\ P_B & P_B & P_B & P_B \end{bmatrix}, \qquad k,m = 0,1,2,3 \quad (10)$$

As in the above example, introducing (10) in (6) and (5), considering (4) we obtain numerically $P_B(4, t_D)$.

Figure 16 shows the agreement with our computer simulations. To generalize the calculation of $A_{k,m}$ we have to distinguish two families of blocking configurations:

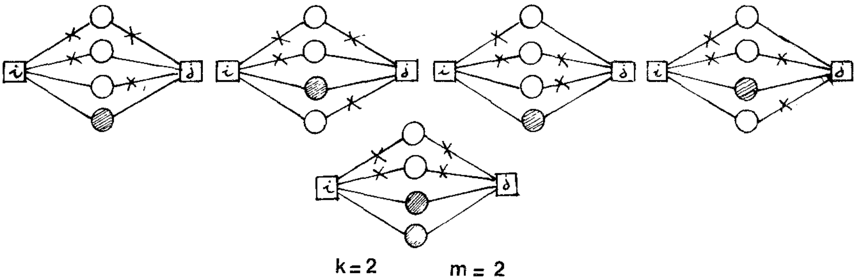

Fig. 10. Blocking configurations described in Fig. 6b. $n = 4$, k input links and m output links are busy.

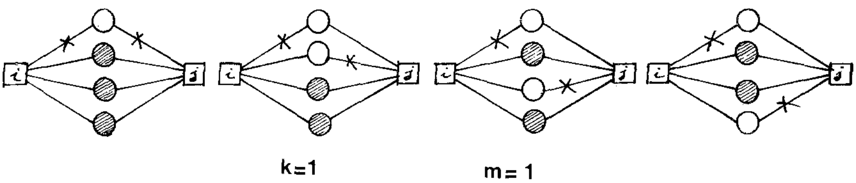

Fig. 11. Blocking configurations described in Fig. 6b. $n = 4$, k input links and m output links are busy.

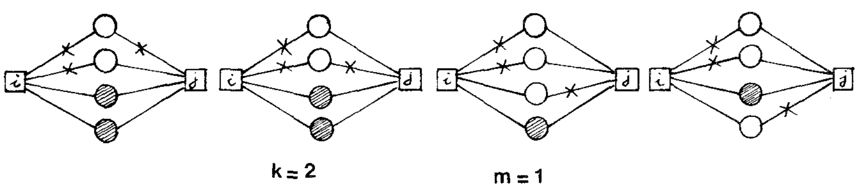

Fig. 12. Blocking configurations described in Fig. 6b. $n = 4$, k input links and m output links are busy.

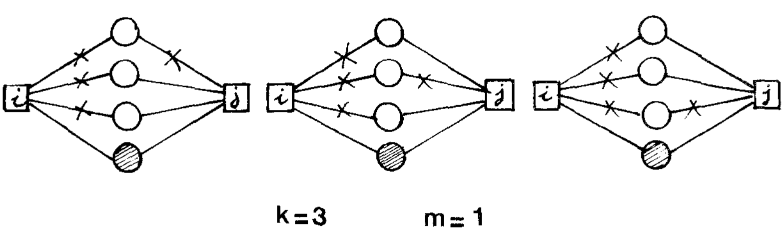

Fig. 13. Blocking configurations described in Fig. 6b. $n = 4$, k input links and m output links are busy.

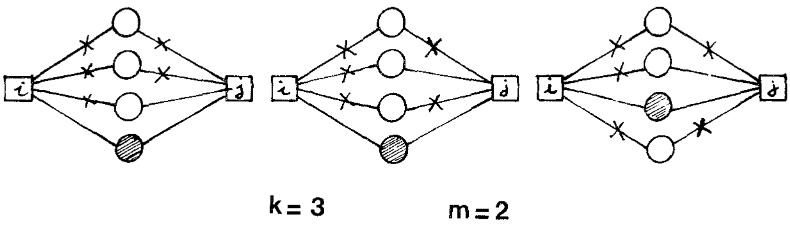

Fig. 14. Blocking configurations described in Fig. 6b. $n = 4$, k input links and m output links are busy.

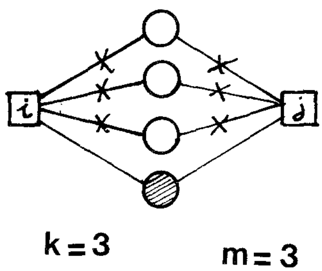

Fig. 15. Blocking configurations described in Fig. 6b. $n = 4$, k input links and m output links are busy.

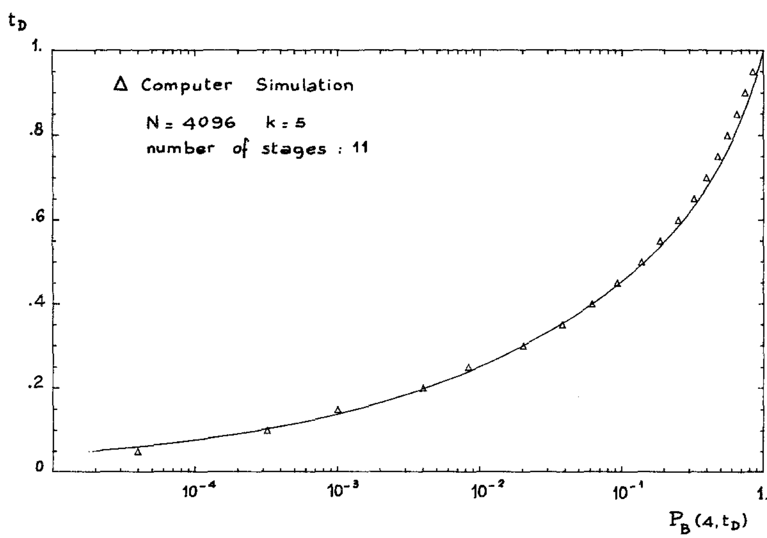

Fig. 16. Blocking probability in the thermodynamic limit. Comparison between the theory and the computer simulations, $n = 4$.

(I) If $k + m \leqslant n - 1$, all the possible blocking configurations are of type described in (b).

(II) If $k + m \geqslant n$, to calculate $A_{k,m}$ we must forget all the possible blocking configurations of type described in (a), induced by mismatches among busy input and output links.

If (I) is true, we have

$$A_{k,m} = \sum_{j=0}^{m} P_B^{n-k-m} \frac{\binom{k}{m-j}\binom{n-k}{j}}{\binom{n}{m}}, \qquad n > k \geqslant m \tag{11a}$$

where $\binom{n}{m} = n!/[(n-m)!m!]$.

If (II) is true, we must subtract from the $\binom{n}{m}$ possible configurations, the $\binom{k}{n-m}$ ones which give up a mismatch of type (a). Consequently,

$$A_{k,m} = \sum_{j=0}^{n-k-1} P_B^{n-k-j} \frac{\binom{k}{m-j}\binom{n-k}{j}}{\binom{n}{m} - \binom{k}{n-m}}, \qquad n > k \geqslant m \tag{11b}$$

The algebraic system (11), (6), (5), and (4) can be solved numerically and gives P_B, the function of the traffic demand t_D.

4. CONCLUSION

The main point in this work is the application of the simple physical concept of thermodynamic limit, where the number k of stages is increased above a critical value so that a computation of the blocking probability for a multistage Clos network is possible, independently of k, by a rather straightforward generalization of the three-stage network.

Notice that from the very beginning, we have supposed that this blocking probability has a limit for $k \to \infty$, in contradiction to the Lee's approach for which this probability goes to 1 for any value of the traffic demand.[5]

We want finally to point out the excellent agreement between the computer simulations and the theoretical results obtained without any introduction of a measured loaded traffic into theoretical formulas.

Our model allows a complete relation between the blocking probability P_B and the two constants μ and λ, which fully characterized the dynamics of the system as realized in our computer simulation.

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